or

$$P\pi r^2\left[(1 + 2\,H/r) + \frac{U(0) + u(0)}{r} + \frac{1}{rP}\int_0^l p \left(\frac{dU}{dx} + \frac{du}{dx}\right) dx\right].$$

The formula for the effective area is now

$$A_P = \pi r^2\left[(1 + 2\,H/r) + \frac{U(0) + u(0)}{r} + \frac{1}{rP}\int_0^l p\left(\frac{dU}{dx} + \frac{du}{dx}\right) dx\right] \qquad (2.2)$$

of which expression (2.1) is a special case with U and u zero. We may evidently obtain (2.2) more directly by visualising the neutral surface as the effective boundary of the piston in which case the frictional force corresponding to b) above vanishes and we are left simply with the pressure forces acting on the effective boundaries of the piston. We also have the equivalent form

$$A_P = \pi r^2\left[1 + 2\,u(0)/r + \frac{2}{rP}\left(\int_0^l -h\,\frac{dp}{dx}\cdot dx + \int_0^l p\,\frac{du}{dx}\cdot dx\right)\right] \qquad (2.3)$$

which is convenient for use when the integral $\int_0^l -h\,\frac{dp}{dx}\,dx$ $\left(\text{or} \int_0^P h\,dp\right)$ is of interest, as is the case, for example, when the flow method (section 5) is considered.

b) The effects of special assumptions

The problem of calculating the actual changes of effective area of practical designs of piston-cylinder assembly, on the basis of the above general formulae, is complicated. It would be necessary to know the interrelated quantities u, U and p as functions of x, and since the pressure gradient dp/dx is governed by the normal equation of viscous flow (see equation 5.1), the pressure dependence of the coefficient of viscosity would also need to be taken into account. It is not, however, the aim of the present paper to attempt such calculations, but rather to describe direct experimental methods for the accurate determination of the distortion factors with the minimum of assumptions regarding the detailed behaviour of the system. We therefore consider only certain special cases which are useful in the applications which follow.

A useful approximation may be derived from the foregoing equations by assuming that the component of $u(x)$ or $U(x)$ due to the fluid pressure in the interspace between piston and cylinder may be taken to be proportional to the pressure $p(x)$ *at the same position.* The relevant terms in the integrals on the right hand side then become integrable without the necessity for any further knowledge of the actual functional forms of $u(x)$, $U(x)$ or $p(x)$. There is fair support from elastic theory for this assumption, more especially in the case of the solid cylinder in which the length is large compared with its radius, a condition which applies to the pistons of most pressure balance assemblies other than those catering for only a low range of pressure. CHREE (1889, 1901) has given polynomial solutions for the equilibrium of a finite solid cylinder for cases in which the lateral pressure is either a linear or quadratic function of the axial co-ordinate. The conditions are satisfied by functions $u(x)$ and $p(x)$ which are accurately proportional, provided the normal tractions over the flat ends, instead of being identically zero, are assumed only to average to zero. By Saint-Venant's principle, however, the effect of this disturbance will be appreciable for only a short distance from each end, and may be neglected if the ratio of length to radius is considerable. The constant of proportionality is the same as in the case of uniform pressure on a solid cylinder of infinite length. FILON (1902) has obtained solutions for pressure distributions expressed in series of trigonometric functions of x which lead to a similar result provided the wavelengths involved are fairly large compared with the radius. The effects of discontinuous pressure distributions, or narrow bands of applied pressure, have also been discussed (BARTON 1941; RANKIN 1944; TRANTER & CRAGGS 1947), with the general result that even the effects of discontinuities are largely lost at an axial distance of only about half the radius. If, therefore, the pressure changes along the length of the assembly are reasonably smooth, no great error is likely to be incurred by applying this assumption to the *piston* of the assembly. Taking into account the additional change of radius due to the end thrust on the piston, it is easily shown that the relevant terms involving u on the right hand side of equation (2.2) reduce to $P\,(3\,\sigma - 1)/2\,E$ where E and σ are respectively Young's modulus and Poisson's ratio, so that we now have, using also (2.1),

$$A_P = A_0\left[1 + \frac{P(3\,\sigma - 1)}{2\,E} + \frac{U(0)}{r} + \frac{1}{rP}\int_0^l p\,\frac{dU}{dx}\cdot dx\right]. \qquad (2.4)$$

Another useful form, obtained directly from (2.3), is

$$A_P = \pi r^2\left[1 + \frac{P(3\,\sigma - 1)}{E} + \frac{2}{rP}\int_0^P h\,dp\right]. \qquad (2.5)$$

The application of a similar assumption to deal with the effects of internal pressure in a hollow cylinder with thick walls is less secure. CHREE (1901) has given a corresponding solution with $U(x)$ and $p(x)$ proportional for the case where $p(x)$ is a linear function of x, but its validity would depend on the conditions assumed at the ends. The case of a discontinuous distribution of pressure has been considered briefly by TRANTER (1946). In the ideal case of a cylinder whose length is large compared with its radius and wall thickness, where the working section is removed some distance from the points of attachment of the ends, and the pressure distribution is reasonably smooth, a useful approximation may result. Proceeding from equation (2.4), and taking for definiteness the case where the cylinder walls are not subjected to longitudinal stress, we then obtain (LOVE 1952), denoting by R' the outer radius of the cylinder,

$$A_P = A_0\left\{1 + \underbrace{\frac{P}{2\,E}(3\,\sigma - 1)}_{\text{(piston)}} + \underbrace{\frac{P}{2\,E}\left[\frac{(1+\sigma)\,R'^2 + (1-\sigma)\,R^2}{R'^2 - R^2}\right]}_{\text{(cylinder)}}\right\}$$